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Abstract. Representations of the canonical and deformed commutation relations by bounded operators on p-adic Banach spaces are constructed. Functions from the Mahler basis of the space of p-adic continuous functions and their multiplicative analogues are shown to be the p-adic counterparts of the Hermite and q-Hermite functions. The analogue of the Stone–von Neumann uniqueness theorem fails in the p-adic case.

It is well known that no bounded operators A, B on a Hilbert space may satisfy the commutation relation $[A, B] = I$ (see, e.g., [1]). The unbounded irreducible representation of this relation (understood in a proper way) is unique up to unitary equivalence within reasonable classes of operators and constitutes one of the main building blocks of quantum mechanics and quantum field theory.

In this paper we show that a p-adic version of this problem possesses quite different features. Note that most of the recent concepts of the p-adic quantum mechanics [2–5] deal with complex-valued wave functions of p-adic arguments, and hence with conventional Hilbert spaces. The problem under consideration arises in quantum mechanics with p-adic valued wavefunctions initiated in [6, 7], and its deeper understanding will hopefully contribute to further development of p-adic methods in physics.

Let p be a prime number, Q_p the field of p-adic numbers, $\mathbb{Z}_p$ the ring of p-adic integers (here and below we use standard notions and notations of p-adic analysis; see, e.g., [2, 8]). Denote by $C(\mathbb{Z}_p, Q_p)$ the Banach space of continuous functions on $\mathbb{Z}_p$ with values in Q_p equipped with sup-norm. The sequence of functions

$$P_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!} \qquad n \geqslant 1 \qquad P_0(x) \equiv 1$$

forms an orthonormal basis of $C(\mathbb{Z}_p, Q_p)$ [9, 10]. This means that every function $f \in C(\mathbb{Z}_p, Q_p)$ admits a unique uniformly convergent expansion

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \qquad c_n \in Q_p$$

with $|c_n|_p \to 0$, and $\|f\| = \sup_n |c_n|_p$. Here $|\cdot|_p$ denotes the absolute value of a p-adic number.

Let us consider on $C(\mathbb{Z}_p, Q_p)$ the operators

$$(a^+ f)(x) = xf(x-1) \qquad (a^- f)(x) = f(x+1) - f(x) \qquad x \in \mathbb{Z}_p\,. \tag{1}$$

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It follows from the non-Archimedean property of the absolute value and the norm that $a^{\pm}$ are defined correctly and $\|a^{\pm}\| \leqslant 1$. Calculating the commutator we easily find that

$$[a^{-}, a^{+}] = I . \tag{2}$$

Simple calculations also show that

$$a^{-}P_n = P_{n-1} \quad n \geqslant 1 \qquad a^{-}P_0 = 0$$
$$a^{+}P_n = (n+1)P_{n+1} \qquad n \geqslant 0$$

so that $a^{\pm}$ are clear analogues of the creation and annihilation operators. Next, putting $H = a^{+}a^{-}$ so that

$$Hf(x) = x\{f(x) - f(x-1)\}$$

we come to an operator with the properties

$$HP_n = nP_n \quad n \geqslant 0 \qquad \left[H, a^{\pm}\right] = \pm a^{\pm} .$$

Note that although the operator H has a complete system of eigenvectors and its point spectrum coincides with the set $\mathbb{Z}_+$ of non-negative integers, the whole spectrum of H equals $\mathbb{Z}_p$, that is the closure of $\mathbb{Z}_+$ in Q_p.

The density of $\mathbb{Z}_+$ in $\mathbb{Z}_p$ implies also that the kernel of a^{-} consists of constant functions. Using this fact and the standard argument from, e.g., [11], we find that a^{-}, a^{+} form an irreducible couple.

Every Banach space over Q_p having an infinite countable orthonormal basis is isomorphic to $C(\mathbb{Z}_p, Q_p)$. Thus the p-adic analogue of the harmonic oscillator Hamiltonian constructed in [6] as a result of a complicated integration theory is in fact equivalent to our much simpler model (since both operators possess orthonormal eigenbases with the same eigenvalues).

It is also not difficult to compute the operator $a^{-}a^{+}$. In particular, $a^{-}a^{+}P_n = (n+1)P_n$, so that $\{P_n\}$ is an orthonormal eigenbasis of $a^{-}a^{+}$.

A small modification of the above construction shows the non-uniqueness of our representation. Namely, let us take instead of a^{-} the operator

$$(a'f)(x) = f(x+1)$$

so that $a' = a^{-} + I$. Of course, $[a', a^{+}] = I$. Consider the operator $H' = a^{+}a'$. We have $(H'f)(x) = xf(x)$ so that H' has no eigenvectors in $C(\mathbb{Z}_p, Q_p)$ and is not equivalent to H in any reasonable sense.

The problem of the complete characterization of irreducible representations remains open. Its solution will probably require further development of p-adic spectral theory [12].

The basis $\{P_n\}$ called the Mahler basis plays a significant role in p-adic analysis. Other objects related to the representation (1) are also well known. For example, the coherent states (eigenfunctions of the annihilation operator a^{-}) are precisely the functions $f_\lambda(x) = (1+\lambda)^x$, $x \in \mathbb{Z}_p$, where $|\lambda|_p < 1$. The proof of this fact immediately follows from the results of [10] where the definition of the function f_λ can also be found.

Another example is a version of the Bargmann–Fock representation realized in the Banach space of power series

$$\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n \qquad z \in \mathbb{Z}_p \tag{3}$$

with coefficients $\varphi_n \in Q_p$, $|\varphi_n|_p \to 0$, and the norm $\|\varphi\| = \sup_n |\varphi_n|_p$ (see [10]). As usual, we may set

$$(b^-\varphi)(z) = \varphi'(z) \qquad (b^+\varphi)(z) = z\varphi(z)$$

with $[b^-, b^+] = I$. Note that in contrast to the classical situation the power series (3) are not necessarily entire functions; they have to be defined only on $\mathbb{Z}_p$. Some other versions of a p-adic Bargmann–Fock representation have been given in [13, 14].

Now let us turn to the deformed commutation relation

$$a_q^- a_q^+ - q a_q^+ a_q^- = I \tag{4}$$

where $q \in Q_p$, $|q|_p = 1$, $q^N \neq 1$ for any $N \in \mathbb{Z}$. In purely algebraic terms such a relation over an arbitrary field was studied in [15]. We are interested in constructing a representation of (4) by bounded operators on a p-adic Banach space.

Let G_q be a closure of the multiplicative cyclic subgroup of the ring $\mathbb{Z}_p$ generated by q. The sequence $\{q^{-n},\ n > 0\}$ is dense in G_q [16]. The explicit description of G_q for some special cases is given in [10]. For example, if $p \neq 2$, $|q-1|_p = p^{-1}$, then $G_q = 1 + p\mathbb{Z}_p$.

Denote by $C(G_q, Q_p)$ the Banach space of all continuous functions on G_q with values in Q_p. An orthonormal basis of this space may be constructed as follows [10]:

$$P_n^{(q)}(x) = \frac{R_n^{(q)}(x)}{R_n^{(q)}(q^{-n})} \quad n \geqslant 1 \qquad P_0^{(q)}(x) \equiv 1 \tag{5}$$

where

$$R_n^{(q)}(x) = (x-1)(x-q^{-1})\cdots(x-q^{-n+1}) \qquad n \geqslant 1\,.$$

A representation of relation (4) by bounded operators on $C(G_q, Q_p)$ is given by

$$\begin{aligned} \left(a_q^+ f\right))(x) &= (x-1)f(qx) \\ \left(a_q^- f\right)(x) &= q(1-q)^{-1}x^{-1}\{f(q^{-1}x) - f(x)\}\,. \end{aligned} \tag{6}$$

Calculating the action of the operators (6) upon the basis (5) we find that

$$\begin{aligned} & a_q^+ P_n^{(q)} = (q^{-n-1} - 1)P_{n+1}^{(q)} \qquad a_q^- P_n^{(q)} = q^n(1-q)^{-1}P_{n-1}^{(q)} \\ & \left(a_q^- a_q^+\right) P_n^{(q)} = (q^n + q^{n-1} + \cdots + 1)P_n^{(q)} \quad n \geqslant 1 \qquad a_q^- a_q^+ P_0^{(q)} = P_0^{(q)} \end{aligned}$$

so that $\left\{P_n^{(q)}\right\}$ is an orthonormal eigenbasis for the operator $a_q^- a_q^+$.

If we introduce an operator N_q by the relation $N_q P_n^{(q)} = n P_n^{(q)}$ ($\|N_q\| \leqslant 1$ since $|n|_p \leqslant 1$) we find that

$$\left[N, a_q^\pm\right] = \pm a_q^\pm\,.$$

Thus we have obtained a p-adic version of the deformed oscillator algebra (see, e.g., [17]; for some related work in the p-adic setting see [18, 19]).

Just as in the conventional quantum mechanics, the existence of a 'vacuum vector' (like P_0 or $P_0^{(q)}$) and the assumption of a 'Hermitian' property of the operator a^-a^+ ($a_q^-a_q^+$) imply the essential features of p-adic representations.

Suppose that $a_q^{\pm}$ are bounded operators on a p-adic Banach space E satisfying relation (4) with $q \in Q_p$ (here we do not exclude the case $q = 1$). Assume that $a_q^-\varphi_0 = 0$ for some $\varphi_0 \in E$, and that the eigenvectors corresponding to different eigenvalues of $a_q^-a_q^+$ (if they exist) are orthogonal (in the p-adic sense). By the induction argument we have

$$a_q^- \left(a_q^+\right)^m - q^m \left(a_q^+\right)^m a_q^- = (q^{m-1} + q^{m-2} + \cdots + 1) \left(a_q^+\right)^{m-1} \qquad m \geqslant 1\,. \tag{7}$$

If the couple $a_q^{\pm}$ is irreducible then it follows from (7) that the vectors $\left(a_q^+\right)^{m-1}\varphi_0,\ m = 1, 2, \ldots,$ form an orthogonal eigenbasis of the operator $a_q^-a_q^+$ in E. The corresponding eigenvalues are $q^{m-1} + q^{m-2} + \cdots + 1$ so that necessarily $|q|_p \leqslant 1$. After a suitable renormalization we obtain the expressions for the action of the operators $a_q^{\pm}$ upon the basis vectors similar to those found above for our function space representations.

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